ON BRITTLE CRACKS UNDER LONGITUDINAL SHEAR

(O KHRUPKIKH TRESHCHINAKH PRODOL'NOGO SDVIGA)

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Some interesting problems on traveling cracks under longitudinal shear were treated in [1] and [2]. Fracture under longitudinal shear presents considerable interest because the mathematical description of this mode of cracking is considerably simpler than in the plane theory of elasticity. For fracture under longitudinal shear it is possible to obtain useful exact solutions of many problems which are inaccessible for cracking in normal fracture and cracks under transverse shear, and thereby certain qualitative effects common to all types of fracture are clarified. Moreover, in problems of longitudinal-shear cracks the accuracy of approximate methods can be conveniently assessed.

The general formulation of the problem of fracture under longitudinal shear is considered below, along with some particular static and dynamic problems.

1. General relations. 1. We assume that the elastic displacements in the body under consideration are such that

$$u, \quad v \equiv 0, \quad w = w(x, y, t) \tag{1.1}$$

where x, y are Cartesian coordinates, t is time, u, v, w are the components of the displacement vector along the axes x, y, z. In view of Hooke's law, the components of the stress tensor are

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \qquad \tau_{yz} = \mu \frac{\partial w}{\partial y}, \qquad \tau_{xz} = \mu \frac{\partial w}{\partial x} \tag{1.2}$$

where $\mu = E/2(1 + \nu)$ is the shear modulus, E Young's modulus and ν Poisson's ratio. Substituting (1.1) into the equations of motion we obtain

$$\frac{\partial^2 w}{\partial t^2} = c^2 \triangle w, \qquad c^2 = \frac{\mu}{\rho} \tag{1.3}$$

where c is the speed of propagation of transverse waves, ρ is the density, and Δ is the Laplace operator. In particular, in the static problem (1.3) reduces to Laplace's equation

$$\Delta w = 0 \tag{1.4}$$

The displacements (1.1) correspond to the case of the so-called "nonplanar" deformation. By nonplanar deformation we mean that the state of stress in a cylindrical body of infinite height arises under the action of loads directed along the generators of the cylinder and constant along the generators.

2. From (1.2) and (1.4) it follows that the stresses and displacements may be represented in terms of a single analytic function $f(z)$ of the complex variable $z = x + iy$; we have

$$w = \operatorname{Re} f(z), \qquad \tau = \tau_{xz} + i\tau_{yz} = \mu \overline{f'(z)} \tag{1.5}$$

There is an obvious analogy between the problem of nonplanar deformation in elasticity theory and the problem of plane hydrodynamics; the displacement w corresponds to the velocity potential, and the stress vector τ corresponds to the velocity vector. This analogy enables one to make use of a number of relations of plane hydrodynamics in the theory of fracture.

In particular, it can be said that for a body bounded by an outer contour c_0 and inner contours $c_1, \ldots, c_n$, the magnitude of the resultant force acting on an arbitrary arc AB is equal to

$$R = \mu \operatorname{Im} [f(z_B) - f(z_A)]$$

In this case the analytic function $f(z)$ can be represented in the form

$$f(z) = \sum_{k=1}^{n} \frac{F_k + i\mu B_k}{2\pi\mu} \ln (z - a_k) + \varphi(z), \qquad \sum_{k=0}^{n} F_k = 0 \tag{1.6}$$

where F_k is the magnitude of the resultant force acting on the contour c_k, B_k is the intensity of the "screw dislocation" corresponding to the contour c_k, i.e. the increment in displacement around the contour c_k, ϕ denotes an analytic function, and a_k is a point of the interior contour c_k.

If the body is unbounded, then in the neighborhood of $z = \infty$ the function $f(z)$ can be represented as follows:

$$f(z) = \frac{F + i\mu B}{2\pi\mu} \ln z + f_0 z + \frac{f_1}{z} + O\left(\frac{1}{z^2}\right) \tag{1.7}$$

where

$$F = \sum_{k=1}^{n} F_k, \quad B = \sum_{k=1}^{n} B_k, \quad f_0 = \frac{1}{\mu}(\tau_{xz}^{\infty} - i\tau_{yz}^{\infty})$$

τ_{xz}^{∞}, τ_{yz}^{∞} being the stresses at infinity.

For the most part we shall consider below the case in which dislocations are absent ($B_k = B = 0$), which corresponds in the hydrodynamic analogy to flow without circulation.

3. Assuming the correctness of the hypothesis that the end region of the crack is small and behaves independently, the authors have previously shown [3] that at those points on the contour of a crack under longitudinal shear where the intensity of the cohesive force is a maximum, the stress τ_{yz}, calculated without taking into account the cohesive force, goes to infinity according to the law

$$\tau_{yz} = \frac{M}{\pi \sqrt{s}} \tag{1.8}$$

where s is the distance to a point on the contour, and M is a material constant, analogous to the cohesive modulus.

2. The simplest problems of cracks under longitudinal shear.

1. Let an infinite body be in a state of nonplanar deformation with shearing stress $\tau_{\infty} = \tau_{\infty}e^{i\theta}$ of constant magnitude at infinity. In the body there is a hole of arbitrary shape but finite dimensions, the surface of which is traction-free. Such a problem corresponds to the problem in two-dimensional hydrodynamics of the flow of an ideal fluid about a contour without circulation. According to well-known relations [5] we have in this case

$$f(z) = \frac{1}{\mu}\tau_{\infty}e^{-i\theta}g(z) + \frac{\tau_{\infty}e^{i\theta}R^2}{\mu g(z)} \tag{2.1}$$

where $g(z)$ is the function which maps conformally the exterior of the contour in the physical plane z onto the exterior of the circle of radius R, such that $g'(\infty) = 1$.

We consider as an example the case where the hole is a circle with one or two identical cracks (Fig. 1) perpendicular to the stress vector τ_{∞} at infinity, the direction of which we take to be the y-axis ($\theta = \pi/2$).

In these cases the mapping functions $g(z)$ are written in the respective forms

$$g(z) = \frac{1}{2}Z - \frac{L-r}{2} + \sqrt{\left[\frac{1}{2}Z - \frac{L-r}{2}\right]^2 - \frac{(L+r)^2}{4}} \tag{2.2}$$

$$g(z) = \frac{1}{2} Z + \sqrt{\frac{1}{4} Z^2 - L^2} \tag{2.3}$$

where

$$Z = z + \frac{r^2}{z}, \qquad L = \frac{1}{2}\left(r + l + \frac{r^2}{r + l}\right) \tag{2.4}$$

The conditions determining the dimension l of the dynamically equilibrated crack have, in view of (1.5), (1.8), (2.1) to (2.4), the respective forms

$$[(1 + \lambda)^4 - 1]\,(1 + \lambda)^{-3/2}(2 + \lambda)^{-3/2}\lambda^{-1/2} = \frac{M}{\pi\tau_\infty \sqrt{r}} \qquad \left(\lambda = \frac{l}{r}\right) \tag{2.5}$$

$$\frac{1}{\sqrt{2}} \sqrt{(1 + \lambda)\,[1 - (1 + \lambda)^{-4}]} = \frac{M}{\pi\tau_\infty \sqrt{r}} \tag{2.6}$$

In particular, as $\lambda \to \infty$ we obtain the asymptotic formulas

$$l = \frac{M^2}{\pi^2 \tau_\infty{}^2}, \qquad l = \frac{2M^2}{\pi^2\tau_\infty{}^2} \tag{2.7}$$

corresponding to an isolated crack of length $2l$ and a crack of length l limited at one end in a uniform stress field (the analog of the Griffith crack). As $\lambda \to 0$ we obtain in both cases the asymptotic formula

$$l = \frac{M^2}{2\pi^2\tau_\infty{}^2} \tag{2.8}$$

which agrees with the formula for half the length of a symmetrical dynamically equilibrated crack in an infinite body in a stress field $2\tau_\infty$, which corresponds to the maximum stress concentration near a circular hole.

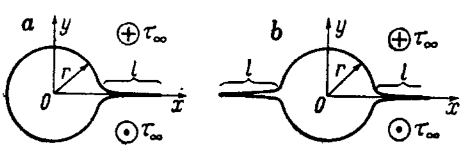

Fig. 1.

The dependence of the length λ of the crack on the applied stress τ_0 is presented in Fig. 2 as a curve of

$$\tau_0 = f(\lambda) \qquad \left(\tau_0 = \frac{\pi\tau_\infty \sqrt{r}}{M}, \quad \lambda = \frac{l}{r}\right)$$

Curves *1* and *2* were obtained from Formulas (2.5), (2.6) and the dashed curves *1′* and *2′* from the asymptotic formulas (2.7).

The analogous problems for cracks in normal fracture were treated in an approximate manner by Bowie [6]; the exact solutions of these problems were not obtained, in view of the irrationality of the mapping functions (2.2), (2.3).

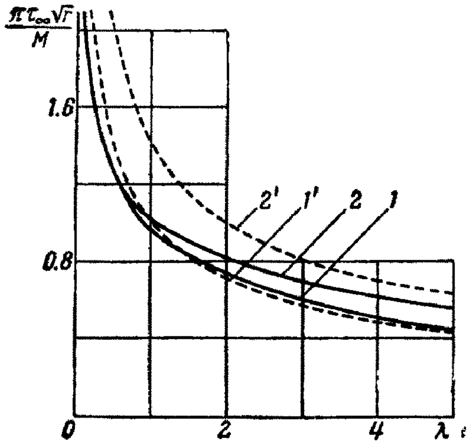

Fig. 2.

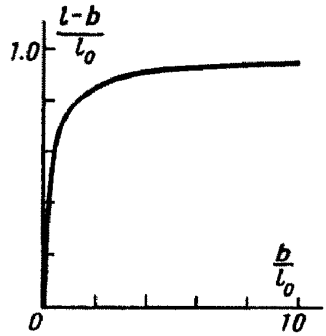

Fig. 3.

2. As an example of a mixed problem we consider the problem of an isolated straight line crack ($-l \leqslant x \leqslant l$), on some portion of the surface of which ($-b \leqslant x \leqslant b$) a constant displacement $w = \pm h$ is prescribed (the plus and minus signs denote the upper and lower faces of the crack). The remainder of the surface of the crack is traction-free. This problem for a normal fracture crack corresponds to the problem of splitting by a wedge of finite length, which was considered in [7]. For the determination of the function $f(z)$ we obtain, obviously, the following boundary value problem:

$$|x| < b, \quad \operatorname{Re} f(z) = \pm h; \qquad b < |x| < l, \quad \operatorname{Im} f'(z) = 0$$
$$|x| > l, \qquad \operatorname{Re} f(z) = 0 \tag{2.9}$$

We find

$$f'(z) = \frac{hl}{i\sqrt{(z^2 - l^2)(z^2 - b^2)}\, F\left(\sqrt{1 - \left(\frac{b}{l}\right)^2}, \frac{\pi}{2}\right)} \tag{2.10}$$

where $F(k, \pi/2)$ is the complete elliptic integral of the first kind. From the condition (1.8) we obtain the relation

$$(l^2 - b^2)\, F^2\left(\sqrt{1 - \frac{b^2}{l^2}}, \frac{\pi}{2}\right) = \frac{\pi^2\mu^2}{2M^2} h^2 l \tag{2.11}$$

which determines the crack length l. In particular, for $b \to \infty$ we obtain the analog of the solution for a semi-infinite wedge

$$l_0 = \frac{\mu^2 h^2}{M^2} \tag{2.12}$$

After transformation the relation (2.11) may be reduced to the form

$$\frac{l-b}{l_0} = f\left(\frac{b}{l_0}\right)$$

This relation is presented in Fig. 3.

3. Some problems on the interaction of cracks in longitudinal shear. Problems of the interactions of cracks present considerable mathematical difficulty; for cracks under longitudinal shear, in contrast to normal fracture cracks and transverse shear cracks, it is possible to obtain useful exact solutions of many interaction problems. Typical problems concerning one-row cracks lattices are considered below.

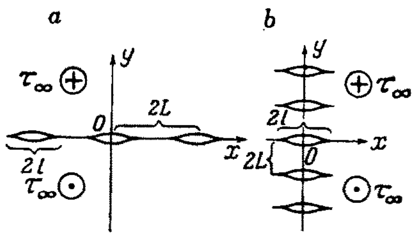

Fig. 4.

1. First of all, suppose that in an infinite body under the action of uniform shear stresses $\tau_{yz} = \tau_{yz}^{\infty}$, at infinity there is an infinite single row of identical cracks $-l + 2nL \leqslant x \leqslant l + 2nL$ $(n = 0, \pm 1, \pm 2, \ldots)$, $y = 0$ (Fig. 4a), perpendicular to the stress vector at infinity, the faces of the cracks being traction-free.

We obtain the following boundary-value problem for the determination of the function $f'(z)$:

$$\lim_{y\to\infty} f'(z) = -\frac{i\tau_{yz}^{\infty}}{\mu}; \ \operatorname{Im} f'(z) = 0, \ -l + 2nL \leqslant x \leqslant l + 2nL, \quad y = 0$$

$$\operatorname{Re} f'(z) = 0, \quad l + 2nL \leqslant x \leqslant -l + 2(n+1)L, \quad y = 0 \tag{3.1}$$

for the solution of which we find

$$f'(z) = -\frac{i\tau_{yz}^{\infty}}{\mu}\frac{\sin(\pi z/2L)}{\sqrt{\sin^2(\pi z/2L) - \sin^2(\pi l/2L)}} \tag{3.2}$$

From Expressions (3.2) and (1.8) we obtain the relation determining the dimension of the cracks for dynamic equilibrium

$$l = \frac{2L}{\pi}\tan^{-1}\frac{M^2}{\pi\tau_{\infty}^{2} L} \tag{3.3}$$

2. We now treat the case where identical cracks form a one-row lattice $-l \leqslant x \leqslant l$, $y = nL$, $n = 0, \pm 1, \pm 2, \ldots$ (Fig. 4b) in an infinite body which is subjected to uniform shear stress $\tau_{yz} = \tau_{yz}^{\infty}$, $\tau_{xz} = 0$ at infinity.

Solving the corresponding boundary-value problem, we obtain

$$f'(z) = -\frac{i\tau_{yz}^{\infty}}{\mu} \frac{\sinh(\pi z / 2L)}{\sqrt{\sinh^2(\pi z/2L) - \sinh^2(\pi l / 2L)}} \tag{3.4}$$

From Expressions (3.4) and (1.8) we obtain the relation determining the dimension l of the dynamically equilibrated cracks

$$l = \frac{2L}{\pi} \tan^{-1} \frac{M^2}{\pi \tau_{\infty}^{2} L} \tag{3.5}$$

The dependence of τ_{∞} on l is represented in Fig. 5 in the form of curves of

$$\frac{\tau_{\infty}}{\tau^*} = f\left(\frac{l}{L}\right) \qquad \left(\tau^* = \frac{M}{\sqrt{\pi L}}\right)$$

Curve *2* was drawn for an isolated crack, from Formula (2.7); curve *3* was drawn for a system of colinear cracks, from Formula (3.3); curve *1* corresponds to a system of parallel cracks, according to Formula (3.5). Clearly, the interaction of cracks is considerably different for various configurations. The presence of colinear cracks diminishes the strength of the body, decreasing the crack size for dynamic equilibrium for a given load. The presence of parallel cracks, on the other hand, strengthens the body, increasing the crack size for dynamic equilibrium for a given load. Moreover, for the parallel cracks there exists a limiting load equal to

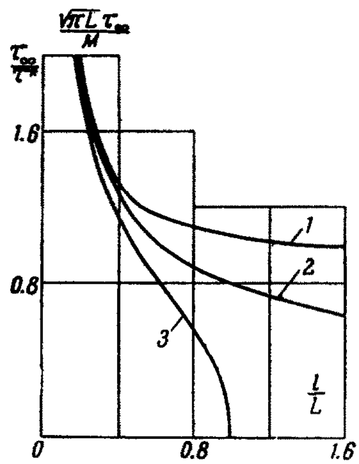

Fig. 5.

$$\tau^* = \frac{M}{\sqrt{\pi L}} \tag{3.6}$$

so that for $\tau_{\infty} < \tau^*$, since there is no crack length, they cannot be in dynamic equilibrium. The problem in classical elasticity theory of an infinite system of colinear cracks for the case of normal fracture was treated by Westergaard [8] and independently by Koiter [9]. Koiter [10] treated the problem of an infinite system of parallel cracks in transverse shear by an approximate method.

4. Curvilinear cracks in longitudinal shear. 1. We shall consider the region near the end of an arbitrary curvilinear crack (Fig. 6) in longitudinal shear under the action of an arbitrary stress field which causes nonplanar deformation. The analysis of the stress field in the neighborhood of the point O shows that the stress τ_{xz} does not have a singularity at this point, whereas the stress τ_{yz} has a singularity of the type $z^{-1/2}$, so that the stress $\tau_{z\theta}$ (Fig. 6) for small r is expressed in the form

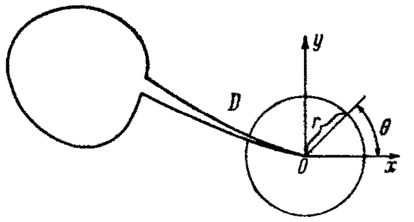

Fig. 6.

$$\tau_{z\theta} = \frac{A_1 \cos(\theta/2)}{\sqrt{r}} + A_2 \sin\theta + O(r^{3/2}) \qquad (4.1)$$

Here A_1 and A_2 are the real coefficients of the first two terms of the expansion of $f(z)$ about $z = 0$:

$$f(z) = \frac{2A_1 i}{\mu}\sqrt{z} + \frac{A_2}{\mu} z + O(z^{3/2}) \qquad (4.2)$$

We make the following hypothesis: *The development of a curvilinear crack under longitudinal shear occurs along the direction in which $\tau_{z\theta}$ is a maximum.*

From this hypothesis it follows that the direction tangent to the surface of a natural crack in longitudinal shear at its end must be the direction of the maximum stress $\tau_{z\theta}$.

In view of (4.1), for this it is necessary and sufficient that

$$A_2 = 0 \qquad (4.3)$$

From (4.1) and (4.3) it follows that the distributions of stress and displacement near the point O are symmetric relative to the direction of the crack. This property of symmetry in the small enables one to make the hypothesis of the independence of the end region, in which the cohesive forces act; i.e. the hypothesis is that the shape of this region and the distribution of cohesive forces in it are independent of the load acting. Assuming the correctness of such a hypothesis regarding the smallness of the end region, we find that at the ends of a longitudinal shear crack, at which the cohesive forces have their maximum intensity, the stresses τ_{yz}, calculated without including the cohesive forces, approach infinity according to the law (1.8), whence

$$A_1 = \frac{M}{\pi} \qquad (4.4)$$

Let $z = \omega(\zeta)$ be a function which maps the exterior of the contour D in the physical plane $z = x + iy$ onto the upper half-plane, and $F(\zeta) = f[\omega(\zeta)]$. Then the conditions (4.3) and (4.4) may be put into the form

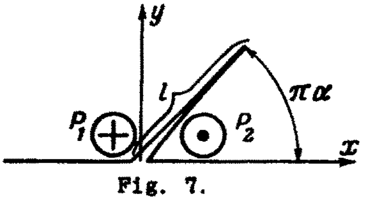

Fig. 7.

$$F''(\zeta_0) = 0 \qquad (4.5)$$

$$\frac{F'(\zeta_0)}{\sqrt{|\omega''(\zeta_0)|}} = \frac{\sqrt{2}M}{\pi\mu} \qquad (4.6)$$

where ζ_0 is the image of the point $z = 0$.

2. We shall now consider some examples. For the first example we take a rectilinear crack which goes out at an angle $\pi\alpha$ to a free surface and is maintained by two oppositely directed concentrated forces P_1 and P_2 acting on different sides of the crack at the point where the crack leaves the free surface (Fig. 7). We have in this case

$$\omega(\zeta) = l(\zeta - 1)^{\alpha}\left(1 + \frac{\alpha}{1-\alpha}\zeta\right)^{1-\alpha} \tag{4.7}$$

Solving the boundary-value problem, we find that the expression for $F'(\zeta)$ in this case is

$$F'(\zeta) = \frac{\alpha}{\pi\mu}\left[\frac{P_1}{1-\alpha+\alpha\zeta} - \frac{P_2}{\alpha(\zeta-1)}\right] \tag{4.8}$$

The conditions (4.3) and (4.4) give

$$\alpha = \frac{\sqrt{P_2}}{\sqrt{P_1}+\sqrt{P_2}}, \qquad l = \frac{\sqrt{P_1P_2}}{2M^2}[\sqrt{P_1}+\sqrt{P_2}]^2 \tag{4.9}$$

Thus the formulation of the problem applies only in the case when the ratio of the forces P_1 and P_2 is constant during the loading process, and consequently the growth of the crack occurs in a straight line only for proportional loading.

Considerable interest attaches to the treatment of a somewhat different problem which is formulated in the following manner. Two symmetrical rectilinear cracks start from the traction-free boundary of a half-space and are held open by concentrated forces applied to the different sides of the cracks at the point where the cracks leave the free surface (Fig. 8a). In this case we have

$$\alpha = \frac{\sqrt{P_1}}{\sqrt{2}(\sqrt{P_2}+\sqrt{2P_1})}, \qquad l = \frac{\sqrt{P_1P_2}}{\sqrt{8}M^2}(\sqrt{P_2}+\sqrt{2P_1})^2 \tag{4.10}$$

Thus for proportional loading the cracks grow along straight lines. As is clear, $\alpha \to 0$ for $P_1 \to 0$, so that the presence of forces P_1 different from zero and directed opposite to the main force P_2 is essential in order that straight line cracks propagate into the interior of the body.

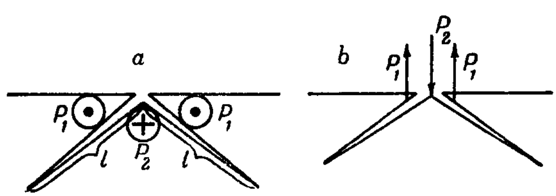

Fig. 8

The above problem for cracks under longitudinal shear will be the analog of the problem of conical cracks in normal fracture, which was treated by Roesler and Benbow [11,12] for the case of an axisymmetric punch impressed on a brittle body. The representation of the effect of the punch by a single concentrated force is insufficient for the correct description of this phenomenon, and it is necessary to introduce oppositely directed concentrated forces (Fig. 8b); otherwise one will not obtain a nonzero included angle for the crack.

3. The formulas given in Section 4.1 in principle enable one to investigate the growth of arbitrary curvilinear cracks for an arbitrary loading process. However, generally speaking, these formulas are not convenient. The effective treatment of the growth of a curvilinear crack is possible for cracks which deviate but little from a straight line or circle. For simplicity the investigation will be restricted to cracks which differ only by a small amount from a straight line crack proceeding at a right angle from a free surface (Fig. 7).

The problem is set up in the following manner. Let both of the applied forces P_1 and P_2 depend on a loading parameter λ so that

$$P_1 = P(\lambda) + \varepsilon P_0(\lambda), \qquad P_2 = P(\lambda)$$

where ϵ is a small number. On account of continuity it may be assumed that the polar coordinate l of the end of the crack and α differ but little from the undisturbed values of these coordinates. Applying the boundary conditions on the y-axis and using the solution of Section 4.2, we find

$$\alpha = \frac{1}{2} - \frac{\varepsilon P_0}{8P}, \qquad l = \frac{2P^2}{M^2}\left(1 + \varepsilon\frac{P_0}{P}\right) \tag{4.11}$$

In particular, for $P_0 = R\theta(\lambda - \lambda_0)$, where R is a constant, θ the unit step function and λ_0 some value of the parameter λ, the coordinates of the end of the crack experience a jump as the parameter λ passes through the value λ_0.

In the case when

$$P_0 = R\sin(\lambda - \lambda_0)\,\theta(\lambda - \lambda_0), \qquad P = \lambda,$$

the crack, starting with a force equal to λ_0, oscillates about the y-axis in a curve, the amplitude of which increases without bound as the force P increases.

5. The dynamic problem of shearing a body. The problem of shearing a body, analogous to the problem of splitting by a normal fracture crack, is formulated in the following manner. A straight line crack propagates with constant velocity V in an unbounded brittle body. The opposite faces of the crack move in opposite directions parallel to the edge of the crack, so that there is a state of nonplanar deformation

(Fig. 9). If we introduce a moving system of coordinates $\xi = x + Vt$, $\eta = y$ with origin at the end O of the crack, then because the process becomes a steady one in the moving coordinates, Equation (1.3) takes the form

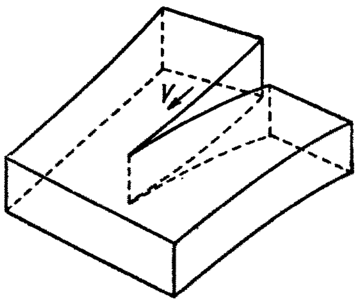

Fig. 9.

$$\frac{\partial^2 w}{\partial \eta^2} + \left(1 - \frac{V^2}{c^2}\right)\frac{\partial^2 w}{\partial \xi^2} = 0 \tag{5.1}$$

The general solution of (5.1) has the form

$$w = \mathrm{Re}\,\varphi(\zeta), \quad \zeta = \xi + i\eta\sqrt{1 - \frac{V^2}{c^2}} \tag{5.2}$$

where $\phi(\zeta)$ is an arbitrary analytic function, so that according to (1.2) the expressions for the stresses have the form

$$\tau_{xz} = \mu \mathrm{Re}\varphi'(\zeta), \qquad \tau_{yz} = -\mu\sqrt{1 - \frac{V^2}{c^2}}\,\mathrm{Im}\,\varphi'(\zeta) \tag{5.3}$$

Exactly as in the problem of splitting [13], the boundary conditions of the problem may be written in the form

$$0 \leqslant \xi \leqslant l, \quad \tau_{\eta z} = 0, \quad l \leqslant \xi < \infty, \quad w = \pm f(\xi) \tag{5.4}$$

where $f(\xi)$ is a given function determining the displacement, which is assumed to be nondecreasing and to approach a finite limit h as $\xi \to \infty$; the plus and minus signs correspond to the upper and lower faces of the crack; l is the length of the free portion of the crack.

Confining ourselves to the case $V < c$, we obtain the following boundary-value problem for the determination of the function $\phi(\zeta)$ in the lower half-plane:

$$\mathrm{Re}\,\varphi(\zeta) = 0, \quad \xi < 0; \quad \mathrm{Im}\,\varphi'(\zeta) = 0, \quad 0 \leqslant \xi < l$$
$$\mathrm{Re}\,\varphi(\zeta) = -f(\xi), \quad l \leqslant \xi < \infty \tag{5.5}$$

Using the formula of Keldysh and Sedov [14], we obtain

$$\varphi'(\zeta) = \frac{1}{\pi i\sqrt{\zeta(\zeta - l)}}\left[\int_l^\infty \frac{f'(t)\sqrt{t(t-l)}\,dt}{t - \zeta} + C\right] \tag{5.6}$$

where the branch of the function $\sqrt{[\zeta(\zeta - l)]}$ is chosen so that $\sqrt{[\zeta(\zeta - l)]} \sim \zeta$ for large ζ. Integrating (5.6) and using the limiting form of the conditions (5.5) for $\xi \to \infty$, we obtain $C = h$. We find for the stress $\tau_{\eta z}$ at $\eta = 0$

$$\tau_{\eta z} = \begin{cases} \dfrac{\mu\sqrt{1-V^2/C^2}}{\pi\sqrt{\xi(\xi-l)}}\left[h - \displaystyle\int_l^\infty \frac{f'(t)\sqrt{t(t-l)}}{t-\xi}\,dt\right] & (-\infty < \xi < 0) \\ 0 & (0 \leqslant \xi < l) \\ -\dfrac{\mu\sqrt{1-V^2/C^2}}{\pi\sqrt{\xi(\xi-l)}}\left[h - \displaystyle\int_l^\infty \frac{f'(t)\sqrt{t(t-l)}}{t-\xi}\,dt\right] & (l \leqslant \xi < \infty) \end{cases} \qquad (5.7)$$

From (5.7) and the condition (1.8), which may be used in the dynamic problem as well, keeping in mind the possible dependence of the quantity M on the velocity V, we obtain the equation for the determination of the free length l of the crack

$$h - \int_l^\infty f'(t)\sqrt{\frac{t-l}{t}}\,dt = \frac{M\sqrt{l}}{\mu\sqrt{1-V^2/c^2}} \qquad (5.8)$$

In the particular case where $f(\xi) \equiv h$, we have

$$l = \frac{\mu^2 h^2}{M^2}\left(1 - \frac{V^2}{c^2}\right) \qquad (5.9)$$

As is clear from (5.9), for longitudinal shear cracks the limiting velocity of propagation is the velocity of sound c in contrast to cracks in normal fracture and transverse shear, for which the limiting velocity is the velocity of propagation of Rayleigh waves.

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